

Scattering in the adjoint sector of the $c = 1$ matrix model

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Scattering in the adjoint sector of the $c = 1$ matrix model

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ABSTRACT: Closed string tachyon emission from a traveling long string in Liouville string theory is studied. The exact collective field Hamiltonian in the adjoint sector of the $c=1$ matrix model is computed to capture the interaction between the tip of the long string and the closed string tachyon field. The amplitude for emission of a single tachyon quantum is obtained in a closed form using the chiral formalism.

KEYWORDS: Matrix Models, Bosonic Strings.

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1. Introduction

As is very well known, the singlet sector of Matrix Quantum Mechanics of a hermitian matrix in an inverted harmonic oscillator potential (known as the $c=1$ matrix model) is equivalent to Liouville string theory in two spacetime dimensions. Liouville conformal field theory amplitudes on closed surfaces are believed to be reproduced by the scattering amplitudes in the MQM (with the genus expansion corresponding to semi-classical expansion of MQM) [1–4]. In contrast, the spacetime and worldsheet interpretation of the non-singlet sectors of the same Matrix Quantum Mechanics is less clear, especially under Lorentzian time signature.¹

In [6] it was proposed that the adjoint sector corresponds to Liouville string theory in the presence of a single long string. In the spacetime picture, the ends of the long string are at infinity, in the asymptotically free region away from the Liouville wall. The string itself is folded in half, and the place where it folds, the ‘tip’, moves under combined effects of string tension and stored kinetic energy. The tip moves nearly with the speed of light towards the Liouville wall until the long string is fully stretched, and then ‘snaps’ back to the free region. Depending on its kinetic energy, it might or might not travel far enough into the bulk of spacetime to ‘hit’ the Liouville wall.

¹Somewhat more is known about the theory in Euclidean time [5]; the intriguing results in [5] are one motivation for exploring the Lorentzian signature further.

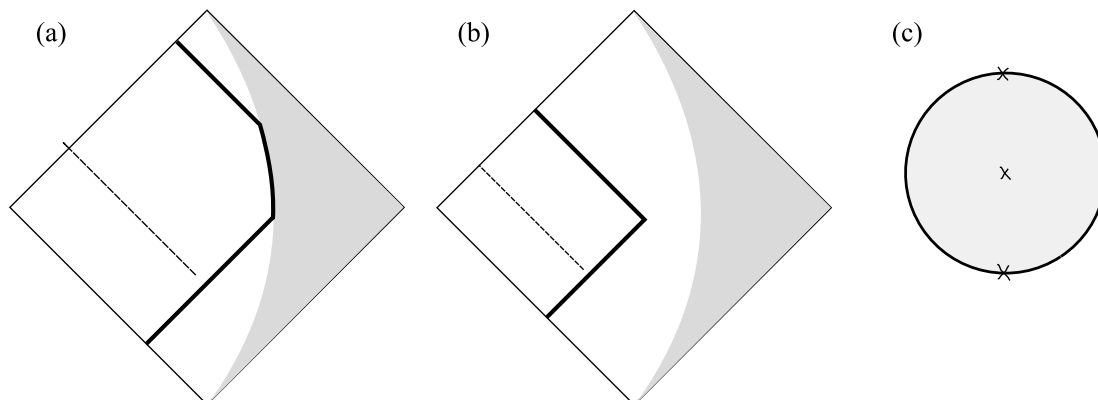


Figure 1: A long string emitting a tachyon. In the Penrose diagrams in (a) and (b), the tip of the long string is drawn as a heavy line, the emitted tachyon is a dashed line, and the shaded region is the region ‘beyond the Liouville wall’ where strings cannot propagate. **(a)** A high energy long string. **(b)** A low energy long string, which does not reach the Liouville wall. **(c)** The corresponding three point disk amplitude. The boundary of the disk must have a large cosmological constant.

The worldsheet interpretation of the adjoint sector requires addition of an open string boundary with two boundary operator insertions corresponding to the incoming and outgoing state of the long string. To keep the ends of the long string at infinity, the boundary can be taken to be an FZZT brane with a very large boundary cosmological constant (as compared to the bulk cosmological constant). The energy of the boundary operators must be large enough to allow for the string to be thought of as ‘long’ (see [6] for details of the limits involved).

This picture allows one to use the worldsheet point of view to study scattering of such long strings. This was done for a single long string in [6] (the scattering amplitude is simply the disk boundary two point function in the appropriate limit). and for multiple long strings in [7]. Worldsheet results were found to agree with matrix model results in [6–10].

The main purpose of the present paper is to extend previous work on scattering of long strings to include backreaction. This should allow us to further test the long string proposal, as well as explore the physics of long strings. In particular, we will be studying emission of closed string tachyons, the only degrees of freedom in Liouville string theory which can travel to null infinity. Emission of a single tachyon quantum is shown in figure 1. This emission can be obtained from a disk diagram with two boundary insertions (for the incoming and outgoing long string) and one bulk tachyon insertion. In the present paper, we will focus on computing this amplitude in the matrix model, and leave the conformal theory calculation to future work. The situation pictured in figure 1(b) is probably easier from the worldsheet point of view, since the bulk cosmological constant can be set to zero.

In Matrix Quantum Mechanics, restricting to the adjoint sector (as opposed to the singlet sector) corresponds to introducing an interacting impurity into the theory of free fermionic matrix eigenvalues. This impurity corresponds to the tip of the long string, and we would like to study its interactions with the background eigenvalue density.

In section 2, we review quantization of Matrix Quantum Mechanics in any sector, including the adjoint. Following this, we study interaction between the adjoint impurity and the rest of the eigenvalues in two ways.

In sections 3–5, we use standard methodology [9, 11, 12] to derive the collective field Hamiltonian describing the interaction between the impurity and the density of eigenvalues. Our result at the end of section 3 extends that obtained in [9], where only quadratic terms in the Hamiltonian were computed. In section 4, we derive the same result using second quantized fermionic description, and bosonization. In section 5, we discuss the form of the collective field Hamiltonian and some of the surrounding technical issues. The results on the collective field Hamiltonian in sections 3 and 4 are applicable to any potential.

In section 6, we derive the amplitude for emission of a single tachyon quantum during long string scattering. We use the chiral formalism [13], adopted for computations involving long strings in [10] and used to describe scattering of multiple long strings in [7]. Our approach here is suitable only for the inverted harmonic oscillator potential. The final result of this calculation, given in equation (6.27), should reproduce the disk amplitude in figure 1(c) once all the appropriate leg-pole factors are taken into the account.

2. Quantization of the matrix model

In this section, for the sake of completeness, we review the quantization of an ungauged one-matrix model with an arbitrary potential V . The classical Lagrangian is

$$\mathcal{L}(M) = \frac{1}{2} \text{tr}(\dot{M})^2 - \text{tr}V(M), \tag{2.1}$$

where M is a hermitian $N \times N$ matrix, while the quantum Hamiltonian can be written simply as

$$H = -\frac{1}{2} \frac{\partial}{\partial M_{ab}} \frac{\partial}{\partial M_{ba}} + \text{tr}V(M), \tag{2.2}$$

when acting on a wavefunction $\Psi(M)$. The lower case Roman indices are matrix indices and run from 1 to N . The kinetic term is just that of N^2 free particles, but the potential couples these in a complicated matter. To take advantage of the (global) $SU(N)$ gauge symmetry present in this problem:

$$SU(N) : M \rightarrow U M U^\dagger, \tag{2.3}$$

it is helpful to decompose M into a diagonal part Λ and an angular part $\Omega \in SU(N)$, $M = \Omega \Lambda \Omega^\dagger$, and then study the wavefunctions in each irrep of $SU(N)$, keeping in mind that the decomposition into Λ and Ω is not unique. We will follow the approach of [14]. Consider a basis of wavefunctions Ψ_α spanning an irreducible representation of $SU(N)$

$$\Psi_\alpha(U M U^\dagger) = U_{\alpha\beta} \Psi_\beta(M). \tag{2.4}$$

The Greek indices are $SU(N)$ representation indices. The Hamiltonian does not couple different irreducible representations to each other.

Consider a point where M is diagonal, $M = \Lambda$. Then, for an action of a diagonal $SU(N)$ matrix D we have

$$\Psi_\alpha(\Lambda) = \Psi_\alpha(D\Lambda D^\dagger) = D_{\alpha\beta}\Psi_\beta(\Lambda) . \quad (2.5)$$

Therefore, $\Psi_\alpha(\Lambda)$ must be a zero-weight vector in the irrep under consideration (transforms trivially under the cartan subgroup). We need then to restrict our attention to irreps with zero-weight vectors. If we denote by $v_\alpha^{(k)}$ a basis for the zero-weight subspace of the representation and by K its dimension, we must have

$$\Psi_\alpha(\Lambda) = \sum_{i=1}^K f^{(k)}(\Lambda)v_\alpha^{(k)} , \quad (2.6)$$

and

$$\Psi_\alpha(M = \Omega\Lambda\Omega^\dagger) = \sum_{k=1}^K f^{(k)}(\lambda_1, \lambda_2, \dots, \lambda_N)\Omega_{\alpha\beta}v_\beta^{(k)} , \quad (2.7)$$

where λ_i are the eigenvalues of M . The problem is therefore reduced to finding the K ‘radial’ functions $f^{(k)}(\lambda_i)$.

Notice that, by construction, this is well defined, since the ambiguity in decomposition of M into Ω and Λ does not affect anything. When Ω is multiplied on the right by a diagonal matrix D , we have

$$(\Omega D)_{\alpha\beta}v_\beta^{(k)} = (\Omega)_{\alpha\beta}(D)_{\beta\gamma}v_\gamma^{(k)} = (\Omega)_{\alpha\beta}v_\beta^{(k)} . \quad (2.8)$$

Another possible ambiguity arises when the eigenvalues are permuted. Given a permutation $\sigma \in S_N$, let P be a permutation matrix $P = \delta_{i,\sigma(i)}$. P is not necessarily in $SU(N)$, as $\det(P) = (-1)^\sigma$, the sign of the permutation is not necessarily 1. We define a matrix in $SU(N)$ as

$$\tilde{P} = ((-1)^\sigma)^{1/N} P = \exp\left(\frac{1 - (-1)^\sigma}{2} \frac{\pi i}{N}\right) P \quad (2.9)$$

Then,

$$(\Omega\tilde{P})_{\alpha\beta}v_\beta^{(k)} = (\Omega)_{\alpha\beta}\tilde{P}_{\beta\gamma}v_\gamma^{(k)} = (\Omega)_{\alpha\beta} \sum_{(k)} \tilde{P}_{(k)(l)}v_\beta^{(l)} , \quad (2.10)$$

where $\tilde{P}_{(k)(l)}$ is the modified permutation matrix in the zero-weight space representation. This means that there is an action on the indices (k) induced by P and we obtain a symmetry constraint on the f 's

$$f^{(k)}(\lambda_{\sigma(i)}) = \sum_{(k)} \tilde{P}_{(k)(l)} f^{(l)}(\lambda_i) . \quad (2.11)$$

It is worth stressing that the origin of the condition (2.11) is due solely to the non-uniqueness of the decomposition $M = \Omega\Lambda\Omega^\dagger$.

Naively, the action of σ on the zero weight space given by (2.11) and (2.9) is badly defined. In particular, let P_1 and P_2 be odd permutations, and $P = P_1P_2$. Then $P = \tilde{P}$ while

$$\tilde{P}_1\tilde{P}_2 = \exp\left(\frac{2\pi i}{N}\right) P_1P_2 . \quad (2.12)$$

In fact, the $\exp(2\pi i/N)$ factor does not cause a problem only because it acts trivially on the zero weight space (the determinant is equal to 1).

Note that for $K = 1$, the unique zero-weight vector is not necessarily invariant under the action of σ . The symmetric group S_N has two one dimensional representations; for the trivial representation, $f(\lambda_i)$ is a totally symmetric function of the eigenvalues, and for the alternating representation, $f(\lambda_i)$ must be totally antisymmetric. For example, for the totally symmetric representation of $SU(N)$ with kN boxes, we have $K = 1$, and the wavefunctions are completely symmetric when k is even, and completely antisymmetric when k is odd.

The allowed representations are those which have at least one zero-weight state. This condition is equivalent to considering only those representations which arise as in the decomposition of tensor powers of the adjoint representation, which in turn is equivalent to requesting that the number of boxes in the Young tableaux must be divisible by N (which allows the diagram to be drawn in the simplified boxes-and-antiboxes form [15]).

To obtain the Schrodinger equation, we need to rewrite the Hamiltonian in terms of the new variables Ω and λ_i . This can be accomplished with the following two formulas²

$$\frac{\partial \Omega_{ki}}{\partial M_{ab}} = \sum_{j \neq i} \frac{\Omega_{ja}^\dagger \Omega_{bi}}{\lambda_i - \lambda_j} \Omega_{kj}, \quad (2.13)$$

$$\frac{\partial \lambda_i}{\partial M_{ab}} = \Omega_{ia}^\dagger \Omega_{bi}. \quad (2.14)$$

A complex conjugate of formula (2.13) is also useful

$$\frac{\partial \Omega_{ik}^\dagger}{\partial M_{ba}} = \sum_{j \neq i} \frac{\Omega_{aj} \Omega_{ib}^\dagger}{\lambda_i - \lambda_j} \Omega_{jk}^\dagger. \quad (2.15)$$

Armed with these formulas, it is possible to explicitly write the action of the Hamiltonian (2.2) on the wavefunction (2.7), once the angular dependence contained in $\Omega_{\alpha\beta}$ is known. While explicit, this calculation is quite cumbersome for even the simplest representations.

A much more efficient way of obtaining the Schrodinger equation in a given representation is to start with the Lagrangian in radial coordinates. Under the decomposition $M = \Omega \Lambda \Omega^\dagger$, the Lagrangian becomes

$$\mathcal{L} = \sum_{i=1}^N \left(\frac{1}{2} \dot{\lambda}_i^2 - V(\lambda_i) \right) + \frac{1}{2} \sum_{i \neq j} (\lambda_i - \lambda_j)^2 \left(\left(\dot{\Omega} \Omega^\dagger \right)_{ij} \right)^2, \quad (2.16)$$

where $\dot{\Omega} \Omega^\dagger = -\dot{\Omega}^\dagger \Omega$ is traceless and anti-hermitian. The classical Hamiltonian can be defined using canonical momenta P_i^λ and P_{ji}^Ω conjugate to λ_i and Ω_{ij} respectively, and

²For completeness, here is a derivation, which is essentially identical with the derivation of first order perturbation theory. Notice that the columns of Ω are eigenvectors of M with eigenvalues λ_i . Consider the equation $M|i\rangle = \lambda_i|i\rangle$, where $|i\rangle$ is the i^{th} column. Varying each side, and hitting the equation with $\langle j|$ on the left, we obtain $\langle j|\delta M|i\rangle + (\lambda_i - \lambda_j)\langle j|(\delta|i\rangle) = (\delta\lambda_i)\delta_{ij}$. The two formulas follow easily from the off-diagonal and diagonal part of this equation respectively.

equals

$$H = \sum_i \left(\frac{1}{2} (P_i^\lambda)^2 + V(\lambda_i) \right) + \frac{1}{2} \sum_{i \neq j} \frac{\Pi_{ij} \Pi_{ji}}{(\lambda_i - \lambda_j)^2}, \quad (2.17)$$

where $\Pi = \Omega(P^\Omega)^T$ are the generators of $SU(N)_R$, the action of $SU(N)$ by multiplication of Ω on the right. Notice that this is not the same as the gauge symmetry of the Hamiltonian, which is $SU(N)_L$.

To understand the meaning and the action of Π_{ij} , let us repeat the derivation of the Hamiltonian (2.17) in a small region around a particular point in angular space, Ω_0 . Write $\Omega(t) = \Omega_0 \exp(iA(t))$, where we will assume that the hermitian matrix A is small, and ignore higher order terms in A . Then, $\dot{\Omega}^\dagger \Omega = \dot{A}$ and the Lagrangian can be written as

$$\mathcal{L} = \sum_{i=1}^N \left(\frac{1}{2} \dot{\lambda}_i^2 - V(\lambda_i) \right) + \frac{1}{2} \sum_{i \neq j} (\lambda_i - \lambda_j)^2 (\dot{A}_{ij})^2. \quad (2.18)$$

It now becomes clear that Π_{ij} is the conjugate momentum to A , $[A_{ij}, \Pi_{kl}] = i\delta_{il}\delta_{jk}$, or

$$\Pi_{ij} = -i \frac{\partial}{\partial A_{ji}}. \quad (2.19)$$

The action of Π on Ω is therefore

$$\Pi_{ij} \Omega_{kl} = -i \frac{\partial}{\partial A_{ji}} (\Omega e^{iA})_{kl} \Big|_{A=0} = \delta_{il} \Omega_{kj} \quad (2.20)$$

and, similarly,

$$\Pi_{ij} \Omega_{kl}^\dagger = -\delta_{jk} \Omega_{il}^\dagger. \quad (2.21)$$

With these formulas on hand, we can use the Hamiltonian in (2.17) to derive the Schrodinger equation in any representation, as long as we take into account the Jacobian for our change of coordinates $M \rightarrow \Omega, \Lambda$. This Jacobian is simply the square of the Vandermonde determinant, Δ^2 , where $\Delta(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$. We must rescale the wavefunction by Δ ; therefore, when acting on wavefunctions of the form

$$\tilde{\Psi}_\alpha(\Lambda, \Omega) = \Delta(\lambda) \sum_{k=1}^K f^{(k)}(\lambda_1, \lambda_2, \dots, \lambda_N) \Omega_{\alpha\beta} v_\beta^{(k)}, \quad (2.22)$$

the Hamiltonian is

$$H = \sum_i \left(-\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i) \right) + \frac{1}{2} \sum_{i \neq j} \frac{\Pi_{ij} \Pi_{ji}}{(\lambda_i - \lambda_j)^2}. \quad (2.23)$$

2.1 The singlet representation

The simplest representation to work with is, of course, the singlet, and this has been much studied. Here, we will just restate a few of the salient facts in our notation. We have $\Omega_{\alpha\beta} = 1$ and therefore

$$\Psi_{\text{singlet}}(M) = f_{\text{symm}}(\lambda_i). \quad (2.24)$$

and

$$\tilde{\Psi}_{\text{singlet}}(M) = \Delta(\lambda) f_{\text{symm}}(\lambda_i) \equiv f_{\text{asym}}(\lambda_i) . \quad (2.25)$$

The Hamiltonian acts as

$$H f_{\text{asym}}(\lambda_i) = \sum_{i=1}^N \left[-\frac{1}{2} \left(\frac{\partial}{\partial \lambda_i} \right)^2 + V(\lambda_i) \right] f_{\text{asym}}(\lambda_i) , \quad (2.26)$$

Since the wavefunction $f_{\text{asym}}(\lambda_i)$ is completely asymmetric under permutations of λ s, this result is the well known result that the problem reduces to N noninteracting fermions in the potential V , where each fermion coordinate corresponds to a single eigenvalue.

2.2 The adjoint

This is the best studied non-singlet representation [14], and the focus of the current paper. There are $N-1$ zero-weight vectors and the problem requires solving $N-1$ coupled differential equations.

Wavefunctions in the adjoint are of the form

$$\Psi_{ij} = \sum_{k=1}^N f_k(\lambda_i) \Omega_{ik} \Omega_{kj}^\dagger , \quad (2.27)$$

where we must impose a constraint

$$\sum_{k=1}^N f_k(\lambda_i) = 0 , \quad (2.28)$$

while condition (2.11) implies that

$$f_{\sigma(k)}(\lambda_i) = f_k(\lambda_{\sigma(i)}) . \quad (2.29)$$

There is no minus sign, since the phases acquired by Ω and Ω^\dagger cancel. An arbitrary wavefunction in the adjoint representation, including the Jacobian factor, is then

$$\tilde{\Psi}_{\text{adjoint}} = \Delta(\lambda) \sum_{k=1}^N f_k(\lambda_i) \left(\Omega^\dagger C \Omega \right)_{kk} \quad (2.30)$$

where C is a traceless matrix of complex coefficients.

Notice that equation (2.29) allows us to find any f_k once one of them, say f_1 , is known. f_1 is completely symmetric in $N-1$ eigenvalues $\lambda_2, \dots, \lambda_N$, with λ_1 ‘special’. The presence of this special eigenvalue leads us to the impurity interpretation.

It is easy to check that

$$\Pi_{ij} \Pi_{ji} \left(\Omega^\dagger C \Omega \right)_{kk} = \left(\Omega^\dagger C \Omega \right)_{kk} (\delta_{ik} + \delta_{jk}) - \left(\Omega^\dagger C \Omega \right)_{ii} \delta_{jk} - \left(\Omega^\dagger C \Omega \right)_{jj} \delta_{ik} \quad (2.31)$$

(with no summation implied anywhere in the above equation), and therefore we obtain a set of Schrodinger equations for wavefunctions $\tilde{f}_k(\lambda) = \Delta(\lambda_i) f_k(\lambda_i)$

$$\sum_{i=1}^N \left[-\frac{1}{2} \left(\frac{\partial}{\partial \lambda_i} \right)^2 + V(\lambda_i) \right] \tilde{f}_k(\lambda_i) + \sum_{k' \neq k} \frac{\tilde{f}_k(\lambda_i) - \tilde{f}_{k'}(\lambda_i)}{(\lambda_k - \lambda_{k'})^2} = E \tilde{f}_k(\lambda_i) . \quad (2.32)$$

The double sum term can be thought of as an ‘impurity-hopping’ interaction term.

3. Collective field analysis

The main goal of this section is to extend the analysis in [9] to include the backreaction from the impurity on eigenvalue density. To this end, we will derive the exact (large N) collective field Hamiltonian in the adjoint sector.

Define

$$\psi(x, 0) = \int \frac{dk}{2\pi} e^{-ikx} \text{tr} \left(e^{ikM} \right), \quad (3.1)$$

$$\psi(x, 1) = \int \frac{dk}{2\pi} e^{-ikx} \text{tr} \left(C e^{ikM} \right), \quad (3.2)$$

where C is a traceless hermitian matrix. Notice that $\psi(x, 0)$ is nothing more than the eigenvalue density

$$\psi(x, 0) = \sum_i \delta(x - \lambda_i) = \rho(x). \quad (3.3)$$

Wavefunction (2.27) can be written as

$$\Psi = \text{tr}(C\Psi) = \int dx \psi(x, 1) \Phi(x, \psi(\cdot, 0)), \quad (3.4)$$

where Φ is a function (or, more correctly, a functional) of the variable x and the field $\psi(\cdot, 0)$. In the adjoint sector, then, we have a theory coupling the density of eigenvalues to a single impurity at x .

In [9] the kinetic term of the Hamiltonian (2.2) is rewritten in terms of derivatives with respect to the fields $\psi(\cdot, 0)$ and $\psi(\cdot, 1)$. The result, when acting on the wavefunction (3.4), is

$$\begin{aligned} -\frac{1}{2} \frac{\partial}{\partial M_{ab}} \frac{\partial}{\partial M_{ba}} &= -\frac{1}{2} \sum_{s=0,1} \int dx \omega(x, s) \frac{\partial}{\partial \psi(x, s)} \\ &\quad -\frac{1}{2} \sum_{s,s'=0,1} \int dx \int dy \Omega(x, s; y, s') \frac{\partial^2}{\partial \psi(x, s) \partial \psi(y, s')}, \end{aligned} \quad (3.5)$$

where

$$\omega(x, s) = \frac{\partial^2 \psi(x, s)}{\partial M_{ab} \partial M_{ba}}, \quad (3.6)$$

$$\Omega(x, s; y, s') = \frac{\partial \psi(x, s)}{\partial M_{ab}} \frac{\partial \psi(y, s')}{\partial M_{ba}}. \quad (3.7)$$

Expressions for Ω and ω in terms of the collective fields can be computed explicitly, either by using the formulas for the laplacian acting on functions in the singlet and adjoint representations, or in Fourier space. In the latter method, we use this simple identity

$$\frac{\partial}{\partial M_{ba}} \left(e^{ikM} \right)_{cd} = ik \int_0^1 d\alpha \left(e^{ik\alpha M} \right)_{cb} \left(e^{ik(1-\alpha)M} \right)_{ad}. \quad (3.8)$$

The results are

$$\omega(x, 0) = -2\partial_x \left[\psi(x, 0) \int dy \frac{\psi(y, 0)}{(x-y)} \right] \quad (3.9)$$

$$\begin{aligned} \omega(x, 1) = & -2\partial_x \left[\psi(x, 1) \int dy \frac{\psi(y, 0)}{(x-y)} \right] \\ & - 2 \left[\psi(x, 1) \int dy \frac{\psi(y, 0)}{(x-y)^2} - \psi(x, 0) \int dy \frac{\psi(y, 1)}{(x-y)^2} \right] \end{aligned} \quad (3.10)$$

$$\Omega(x, 0; y, 0) = \partial_x \partial_y (\psi((x+y)/2, 0) \delta(x-y)) \quad (3.11)$$

$$\Omega(x, 0; y, 1) = \partial_x \partial_y (\psi((x+y)/2, 1) \delta(x-y)) \quad (3.12)$$

Notice that we will not need $\Omega(x, 1; y, 1)$, since the wavefunction (3.4) is linear in $\psi(x, 1)$. (This is a good thing, since $\Omega(x, 1; y, 1)$ involves the collective field for two impurities.)

All the integrals in equations (3.9) and (3.10) should be regularized using the Principal Value prescription, which arises when we perform the following exchange of order of integration

$$\begin{aligned} \int_0^1 d\alpha \int dy (ik) e^{i\alpha k(x-y)} G(x, y) &= P.V. \int dy \int_0^1 d\alpha (ik) e^{i\alpha k(x-y)} G(x, y) \\ &= P.V. \int dy \frac{e^{ik(x-y)} G(x, y)}{x-y} \end{aligned} \quad (3.13)$$

where G is some function. The Principal Value prescription will be implicit from now on in this section.

As was already seen in the previous section, the Jacobian J for changing variables from the matrix to its eigenvalues is simply the Vandermonde determinant squared, $(\Delta(\lambda))^2$. Up to a sign, and ignoring a set of measure zero where the eigenvalues coincide, we have

$$\ln(J) = \sum_{i \neq j} \ln(\lambda_i - \lambda_j) = \int dx dy \psi(x, 0) \psi(y, 0) \ln(x-y) . \quad (3.14)$$

In order to arrive at a self-adjoint Hamiltonian, we must perform a similarity transformation, equivalent to rescaling the wavefunction by the square root of J , which leads to the following substitution [9]

$$\frac{\partial}{\partial \psi(x, 0)} \rightarrow \frac{\partial}{\partial \psi(x, 0)} - \frac{1}{2} \frac{\partial \ln J}{\partial \psi(x, 0)} = \frac{\partial}{\partial \psi(x, 0)} - \int dy \frac{\psi(y, 0)}{x-y} . \quad (3.15)$$

Finally, introducing the canonical conjugate to $\psi(x, 0)$,

$$\Pi(x) = -i \frac{\partial}{\partial \psi(x, 0)} . \quad (3.16)$$

and putting all of this together, we get that those terms in the Hamiltonian which involve only $s = 0$ are

$$\begin{aligned} H_{s=0} = & -\frac{1}{2} \int dx \omega(x, 0) \left(\frac{\partial}{\partial \psi(x, 0)} - \frac{1}{2} \frac{\partial \ln J}{\partial \psi(x, 0)} \right) \\ & -\frac{1}{2} \int dx dy \Omega(x, 0; y, 0) \left(\frac{\partial}{\partial \psi(x, 0)} - \frac{1}{2} \frac{\partial \ln J}{\partial \psi(x, 0)} \right) \left(\frac{\partial}{\partial \psi(y, 0)} - \frac{1}{2} \frac{\partial \ln J}{\partial \psi(y, 0)} \right) \\ & + \int dx \psi(x, 0) (V(x) + \mu) , \end{aligned} \quad (3.17)$$

which combine to give the well-known collective field Hamiltonian

$$\begin{aligned}
 H_{s=0} &= \frac{1}{2} \int dx \partial_x \Pi(x) \psi(x, 0) \partial_x \Pi(x) \\
 &+ \int dx \left(\frac{\pi^2}{6} \psi^3(x, 0) + \psi(x, 0) (V(x) + \mu) \right) \\
 &+ \frac{1}{2} \int dx \psi(x, 0) (\partial_x \partial_y \ln |y - x|_{y=x}) .
 \end{aligned} \tag{3.18}$$

We have added a Lagrange multiplier μ to fix the total number of eigenvalues. To obtain the second line in (3.18), it is necessary to use that

$$\int dx g(x) \left[P.V. \int dy \frac{g(y)}{x - y} \right]^2 = \frac{\pi^3}{3} \int dx (g(x))^3 , \tag{3.19}$$

which is easy to prove in Fourier space.

The last line is the 1-loop contribution to the energy [11].

Terms containing the $s = 1$ fields contribute three more terms to the Hamiltonian,

$$\begin{aligned}
 H_{s=1} &= -\frac{1}{2} \int dx \omega(x, 1) \frac{\partial}{\partial \psi(x, 1)} - \int dx dy \Omega(x, 0; y, 1) \frac{\partial^2}{\partial \psi(x, 0) \partial \psi(y, 1)} \\
 &+ \frac{1}{2} \int dx dy \Omega(x, 0; y, 1) \frac{\partial \ln J}{\partial \psi(x, 0)} \frac{\partial}{\partial \psi(y, 1)} ,
 \end{aligned} \tag{3.20}$$

which evaluate to

$$\begin{aligned}
 H_{s=1} &= \int dx dy \frac{\psi(x, 1) \psi(y, 0) - \psi(x, 0) \psi(y, 1)}{(x - y)^2} \frac{\partial}{\partial \psi(x, 1)} \\
 &- \int dx \psi(x, 1) \left(\partial_x \frac{\partial}{\partial \psi(x, 0)} \right) \left(\partial_x \frac{\partial}{\partial \psi(x, 1)} \right) ,
 \end{aligned} \tag{3.21}$$

When acting with $H_{s=1}$ on the wavefunction (3.4), we get

$$\begin{aligned}
 H_{s=1} \Psi &= \\
 &\int dx \psi(x, 1) \left[\int dy \frac{\psi(y, 0)}{(x - y)^2} [\Phi(x, \psi(\cdot, 0)) - \Phi(y, \psi(\cdot, 0))] + \partial_x \Pi(x) (-i \partial_x) \Phi(x, \psi(\cdot, 0)) \right]
 \end{aligned} \tag{3.22}$$

which when added to the singlet sector Hamiltonian (3.18) gives the exact collective field Hamiltonian in the adjoint sector.

4. Free fermions

In this section, we will re-derive the Hamiltonian using second quantized formalism for free fermions to represent the eigenvalues.

Let us define a function $F(x; \underline{\lambda})$ of a coordinate x and the set of eigenvalues $\underline{\lambda}$, completely antisymmetric in those eigenvalues. We will need this function to have the property that

$$F(x, \underline{\lambda})|_{x=\lambda_k} = \Delta(\underline{\lambda}) f_k(\lambda_i) . \tag{4.1}$$

F could be defined as

$$F(x, \underline{\lambda}) = \sum_k [\Delta(\underline{\lambda}) f_k(\lambda_i)] |_{\lambda_k=x} \quad (4.2)$$

but it is not unique (see the next section for a discussion of this).

Another ingredient we need is two auxiliary fermion fields $\Psi_{\uparrow}(x)$ and $\Psi_{\downarrow}(x)$, such that $\{\Psi_i(x), \Psi_j^{\dagger}(y)\} = \delta(x-y)\delta_{ij}$, and a vacuum state $|\mathbf{0}\rangle$ defined by $\Psi_i(x)|\mathbf{0}\rangle = 0$.

Let us now define a state corresponding to the adjoint wavefunction

$$|f^{adj}\rangle \equiv \left(\prod_{i=1}^N \int d\lambda_i \right) \sum_j \Delta(\underline{\lambda}) f_j(\lambda) \Psi_{\downarrow}^{\dagger}(\lambda_1) \dots \Psi_{\downarrow}^{\dagger}(\lambda_{j-1}) \Psi_{\uparrow}^{\dagger}(\lambda_j) \Psi_{\downarrow}^{\dagger}(\lambda_{j+1}) \dots \Psi_{\downarrow}^{\dagger}(\lambda_N) |\mathbf{0}\rangle \quad (4.3)$$

which can be rewritten in terms of F as

$$|f^{adj}\rangle \equiv \int dx \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x) \left(\prod_{i=1}^N \int d\lambda_i \right) F(x; \underline{\lambda}) \left(\prod_{i=1}^N \Psi_{\downarrow}^{\dagger}(\lambda_i) \right) |\mathbf{0}\rangle. \quad (4.4)$$

The term in front, $\int dx \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x)$, is the same for all states, and can be stripped. What we are left with is

$$\left(\prod_{i=1}^N \int d\lambda_i \right) F(x; \underline{\lambda}) \left(\prod_{i=1}^N \Psi_{\downarrow}^{\dagger}(\lambda_i) \right) |\mathbf{0}\rangle, \quad (4.5)$$

which has the interpretation of a state in a theory of a single fermion field, Ψ_{\downarrow} and an impurity. The \uparrow fermions were only a crutch, and disappear in this way of thinking about it.

To effect the stripping of the $\int dx \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x)$ term, we can multiply any state of the form (4.4) on the left by $\Psi_{\downarrow}^{\dagger}(z) \Psi_{\uparrow}(z)$. This leads to

$$\Psi_{\downarrow}^{\dagger}(z) \Psi_{\uparrow}(z) |f^{adj}\rangle = \Psi_{\downarrow}^{\dagger}(z) \Psi_{\downarrow}(z) \left(\prod_{i=1}^N \int d\lambda_i \right) F(x; \underline{\lambda}) \left(\prod_{i=1}^N \Psi_{\downarrow}^{\dagger}(\lambda_i) \right) |\mathbf{0}\rangle, \quad (4.6)$$

which is the same as equation (4.5) except for a factor of $\Psi_{\downarrow}^{\dagger}(z) \Psi_{\downarrow}(z)$ in front, which is nothing more than the fermion density at point z , $\Psi_{\downarrow}^{\dagger}(z) \Psi_{\downarrow}(z) = \rho(z) = \psi(z, 0)$.

For ease of notation, let us define

$$|\underline{\lambda}\rangle \equiv \left(\prod_{i=1}^N \Psi_{\downarrow}^{\dagger}(\lambda_i) \right) |\mathbf{0}\rangle \quad (4.7)$$

and

$$\int d\underline{\lambda} \equiv \left(\prod_{i=1}^N \int d\lambda_i \right). \quad (4.8)$$

The action of the Hamiltonian in this notation can be derived by taking the left hand side of equation (2.32), multiplying by $\Psi_{\downarrow}^{\dagger}(\lambda_1) \dots \Psi_{\downarrow}^{\dagger}(\lambda_{k-1}) \Psi_{\uparrow}^{\dagger}(\lambda_k) \Psi_{\downarrow}^{\dagger}(\lambda_{k+1}) \dots \Psi_{\downarrow}^{\dagger}(\lambda_N) |\mathbf{0}\rangle$, summing over k , integrating over all eigenvalues and rewriting everything in terms of F

instead of the f_k s. Doing this carefully leads to a number of terms,

$$\begin{aligned}
 H|f^{adj}\rangle &= \int dx \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x) \left[-\frac{1}{2} \int dy \Psi_{\downarrow}^{\dagger}(y) \partial_y^2 \Psi_{\downarrow}(y) \right] \left(\int d\lambda \right) F(x; \lambda) |\lambda\rangle \\
 &+ \int dx \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x) \left[\int dy \Psi_{\downarrow}^{\dagger}(y) (V(y) + \mu) \Psi_{\downarrow}(y) \right] \left(\int d\lambda \right) F(x; \lambda) |\lambda\rangle \\
 &+ \frac{1}{2} \int dx \left(\partial_x \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x) - \Psi_{\uparrow}^{\dagger}(x) \partial_x \Psi_{\downarrow}(x) \right) \left(\int d\lambda \right) \partial_x F(x; \lambda) |\lambda\rangle \\
 &+ \int dx \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x) \left(\int d\lambda \right) P.V. \int dy \frac{F(x; \lambda) - F(y; \lambda)}{(x - y)^2} \Psi_{\downarrow}^{\dagger}(y) \Psi_{\downarrow}(y) |\lambda\rangle .
 \end{aligned} \tag{4.9}$$

The Principal Value appears because the sum in equation (2.32) excludes $k = k'$ terms. It has the effect of allowing us to ignore the anticommutator term between $\Psi_{\downarrow}(y)$ and $\Psi_{\downarrow}(x)$, since $x = y$ is not in the domain of integration. It also matches the P.V. prescription in the bosonic treatment.

The first two lines are nothing but the kinetic and potential terms of a non-relativistic noninteracting fermion field Hamiltonian. As is well known, after bosonization, these turn into the terms in equation (3.18) [16, 17].

The last two lines should correspond to equation (3.22). The exchange term is clearly the same, but we need to look more carefully at the third line of (4.9) and the final term in equation (3.22). The difficulty here is that the universal term $\int dx \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x)$ is not in front explicitly. We will strip it off by the method which lead to equation (4.6),

$$\begin{aligned}
 &\Psi_{\downarrow}^{\dagger}(z) \Psi_{\uparrow}(z) \times \frac{1}{2} \int dx \left(\partial_x \Psi_{\uparrow}^{\dagger}(x) \Psi_{\downarrow}(x) - \Psi_{\uparrow}^{\dagger}(x) \partial_x \Psi_{\downarrow}(x) \right) \left(\int d\lambda \right) \partial_x F(x; \lambda) |\lambda\rangle \\
 &= \frac{1}{2} \left(\partial_z \Psi_{\downarrow}^{\dagger}(z) \Psi_{\downarrow}(z) - \Psi_{\downarrow}^{\dagger}(z) \partial_z \Psi_{\downarrow}(z) \right) \left(\int d\lambda \right) \partial_x F(x; \lambda) |\lambda\rangle
 \end{aligned} \tag{4.10}$$

We recognize the multiplier in front as simply $-i$ times the total momentum density of the fermions at point z , which will we denote with $-iJ(z)$. Comparing with equation (4.6), the third line of the Hamiltonian should be interpreted as

$$-i \frac{J(z)}{\rho(z)} \partial_z . \tag{4.11}$$

We wish to show that this is the same as $\partial_z \Pi(z)$ ($-i\partial_z$) in equation (3.22).

From the Hamiltonian in equation (3.18), we have that

$$\frac{\partial}{\partial t} \rho(x) = -\partial_x (\rho(x) \partial_x \Pi(x)) \tag{4.12}$$

The continuity equation implies that

$$\frac{\partial}{\partial t} \rho(x) = -\partial_x J(x) \tag{4.13}$$

and therefore

$$J(x) = \rho(x) \partial_x \Pi(x) , \tag{4.14}$$

as required.

Notice that in deriving the above correspondence we used the bosonic equation of motion arising from the $s=0$ sector alone, and the continuity equation for fermions which is implied by the kinetic term in the first line of (4.9). In other words, our result is the consequence of bosonization in the singlet sector. We could have also used the bosonic equation of motion modified by the terms in the $s=1$ sector, together with a new continuity equation applicable to a fermionic field whose kinetic term is affected by the last line of (4.9), but this would have been unnecessarily complicated, since equation (4.14) is simply an algebraic consequence of bosonization itself and does not depend on the details of the dynamics.

5. Discussion — Collective field approach

We have now derived the collective field Hamiltonian which describes the interaction of the impurity with the eigenvalue density in two different ways.

It is worth mentioning here that the adjoint wavefunction $\Phi(x, \psi(\cdot, 0))$, and its anti-symmetric equivalent $F(x, \underline{\lambda})$, are not unique. For example, we can add any function of the form $\prod_k (x - \lambda_k) G$ to F , as long as $G(x, \underline{\lambda})$ is regular and antisymmetric in the λ s. This ambiguity should not play a role in the large N limit.

The wavefunction is also subject to the adjoint constraint (2.28),

$$\int dx \psi(x, 0) F(x; \underline{\lambda}) = 0 . \tag{5.1}$$

The ambiguity and the constraint would have to be taken into account when solving the equations of motion.

As usual with the $c=1$ matrix model, equations of motion should be solved perturbatively, in the semi-classical expansion, with N strictly infinite. With the upside down harmonic oscillator potential $V(x) = -\frac{1}{2}x^2$, we can treat the semi-classical expansion as an expansion in μ^{-1} . On the string theory side, this corresponds to expansion in the string coupling, since $\mu = g_s^{-1}$ [17]. Weak string coupling corresponds to large μ , and this is the limit we want to work with.

In order to better exhibit the dependence on μ , we will rescale the coordinate x to $\sqrt{\mu}x$. To maintain the shape of Fermi sea in phase space, we must then rescale [2]

$$\psi(\cdot, 0) \rightarrow \sqrt{\mu} \psi(\cdot, 0) . \tag{5.2}$$

The scaling of Π is determined from fixing

$$\int dx \frac{\partial}{\partial \psi(x, 0)} \psi(y, 0) = 1 , \tag{5.3}$$

to be

$$\Pi(x) \equiv -i \frac{\partial}{\partial \psi(x, 0)} \rightarrow \mu^{-1} \Pi(x) . \tag{5.4}$$

The scaling of $\psi(x, 1)$ does not matter, as it cancels out in the Hamiltonian. Canonical commutation relationships require that the combination $\psi(x, 1) \partial / \partial \psi(x, 1)$ scale as

$$\psi(x, 1) \frac{\partial}{\partial \psi(x, 1)} \rightarrow \mu^{-\frac{1}{2}} \psi(x, 1) \frac{\partial}{\partial \psi(x, 1)} \tag{5.5}$$

With this scaling, the functions in (3.9)–(3.12) scale as

$$\begin{aligned}\omega(x, s) &\rightarrow \sqrt{\mu}\omega(x, s), \\ \Omega(x, 0, y, 1) &\rightarrow \mu^{-1}\Omega(x, 0, y, 1), \\ \Omega(x, 0, y, 1)\frac{\partial}{\partial\psi(y, 1)} &\rightarrow \mu^{-2}\Omega(x, 0, y, 1)\frac{\partial}{\partial\psi(y, 1)}.\end{aligned}\tag{5.6}$$

After this rescaling, the leading term in the Hamiltonian is

$$\mu^2 \int dx \left(\frac{\pi^2}{6}\psi^3(x, 0) + (\mu - x^2/2)\psi(x, 0) \right),\tag{5.7}$$

which comes from the $s = 0$ part of the Hamiltonian. The classical solution is the well-known eigenvalue density distribution given by

$$\pi\psi(x, 0) = \pi\phi_0(x) = \sqrt{x^2 - 2},\tag{5.8}$$

where μ is the Fermi level energy. Allowing the density to fluctuate,

$$\psi(x, 0) = \phi_0(x) + \frac{1}{\sqrt{\pi\mu}}\partial_x\eta,\tag{5.9}$$

we can obtain the $o(\mu^0)$ term in the large μ expansion of the Hamiltonian. Let us focus first on that part of the Hamiltonian which contains only $s = 0$ field. The desired term is

$$\frac{1}{2} \int dx \pi\phi_0(x) (P^2 + (\partial_x\eta)^2)\tag{5.10}$$

where $P = -\partial_x\Pi(x)/(\sqrt{\pi\mu})$ is the conjugate momentum to η . Notice that the μ -scaling of η is necessary to obtain this (quite standard) massless scalar kinetic term. Interactions are higher order in μ^{-1} :

$$\mu^{-1} \int dx \sqrt{\pi} \left(\frac{1}{6}(\partial_x\eta)^3 + \frac{1}{2}P^2\partial_x\eta \right).\tag{5.11}$$

Now, let us turn our attention to the terms involving the $s = 1$ field. There are three such terms in the Hamiltonian, and their μ -scalings are

$$\begin{aligned}\int dx \omega(x, 1)\frac{\partial}{\partial\psi(y, 1)} &\rightarrow \mu^0 \int dx \omega(x, 1)\frac{\partial}{\partial\psi(y, 1)}, \\ \int dx dy \Omega(x, 0; y, 1)\frac{\partial^2}{\partial\psi(x, 0)\partial\psi(y, 1)} &\rightarrow \mu^{-2} \int dx dy \Omega(x, 0; y, 1)\frac{\partial^2}{\partial\psi(x, 0)\partial\psi(y, 1)}, \\ \int dx dy \Omega(x, 0; y, 1)\frac{\partial \ln J}{\partial\psi(x, 0)}\frac{\partial}{\partial\psi(y, 1)} &\rightarrow \mu^0 \int dx dy \Omega(x, 0; y, 1)\frac{\partial \ln J}{\partial\psi(y, 0)}\frac{\partial}{\partial\psi(y, 1)}.\end{aligned}\tag{5.12}$$

The first and the third terms are linear in $\psi(x, 0)$. Any term containing $\partial_x\eta$ as opposed to ϕ_0 is therefore suppressed by $1/\mu$. The second term is linear in $\Pi \sim \mu P$, so any term containing P is also suppressed by $1/\mu$. All the terms of order μ^0 are thus those collected in [9] equation (14). Therefore, order μ^0 part of the Hamiltonian involving $s = 1$ fields is [9]

$$\int dx dy \frac{\phi_0(y)\psi(x, 1) - \phi_0(x)\psi(y, 1)}{(x - y)^2} \frac{\partial}{\partial\psi(x, 1)}.\tag{5.13}$$

Notice that this is the Hamiltonian stated in [6], whose eigenfunctions were found in [8].

To next order, as we have seen above, we have self-interactions of η , as well as coupling between η and the position of the impurity, which are the terms we are interested in,

$$\begin{aligned} & \mu^{-1} \int dx dy \frac{1}{\sqrt{\pi}} \frac{\partial_y \eta(y) \psi(x, 1) - \partial_x \eta(x) \psi(y, 1)}{(x-y)^2} \frac{\partial}{\partial \psi(x, 1)} \\ & - \mu^{-1} \int dx \sqrt{\pi} P(x) \psi(x, 1) \left(-i \partial_x \frac{\partial}{\partial \psi(x, 1)} \right). \end{aligned} \quad (5.14)$$

Interactions between the impurity and the collective field are of the same order as three-vertex self-interactions of the collective field. This corresponds to the statement that the three point function of three tachyons on a sphere is of the same order in g_s (g_s^1) as the disk amplitude with two boundary and one bulk tachyon insertions.

6. Scattering in the chiral formalism

While the Hamiltonian of the previous sections is quite enlightening, it is not the best starting point for a calculation of scattering amplitudes. As was pointed out in [10], those are much easier to calculate in the chiral formalism [13]. We will extend the formalism in [10] to include backreaction on the collective field.

Methods presented so far were applicable to any potential. The chiral formalism we are about to introduce is applicable only to the $V = -\frac{1}{2}x^2$ potential, which is the one relevant for $c=1$ string theory. The chiral formalism for the up-side-down harmonic oscillator is similar to the raising and lowering operator formalism for the ordinary harmonic oscillator. We will make a canonical transformation to the light-cone (or chiral) variables

$$M_{\pm} \equiv \frac{M \pm P_M}{\sqrt{2}}, \quad (6.1)$$

where PM is the canonical conjugate to the matrix M . The Hamiltonian in these variables is

$$H = -\frac{1}{2} \text{Tr} (M_+ M_- + M_- M_+) . \quad (6.2)$$

Since equations of motion imply that M_{\pm} evolves like $e^{\pm t}$, in the far past we have $M \sim M_+$ and in the far future $M \sim M_-$. Therefore, wavefunctions written in M_+ coordinates correspond to incoming states and written in M_- coordinates correspond to outgoing states. Since M_+ and M_- are complimentary, these wavefunctions are related by a Fourier transform, and the scattering matrix is given by an inner product

$$\langle \Psi^- | \Psi^+ \rangle = \int dM_+ dM_- e^{i \text{tr}(M_+ M_-)} \sum_{\alpha=1}^{\dim \mathcal{R}} \overline{\Psi_{\alpha}^-(M_-)} \Psi_{\alpha}^+(M_+) . \quad (6.3)$$

From our formalism in section 2, it is clear that the integral $\int dM_+ dM_-$ can be performed separately over the angular degrees of freedom Ω_{\pm} and the eigenvalue degrees of freedom Λ_{\pm} . As before, we have $M_{\pm} = \Omega_{\pm} \Lambda_{\pm} \Omega_{\pm}^{\dagger}$ and $dM_{\pm} = \prod_i \lambda_i^{\pm} \Delta(\lambda^{\pm}) d\Omega_{\pm}$. Therefore

$$\langle \Psi^- | \Psi^+ \rangle = \int d\lambda_i^+ d\lambda_j^- \Delta^2(\lambda^+) \Delta^2(\lambda^-) d\Omega \overline{\Psi_{\alpha}^-(\Lambda_-)} \Omega_{\alpha\beta} \Psi_{\beta}^+(\Lambda_+) e^{i \text{tr}(\Lambda_+ \Omega \Lambda_- \Omega^{\dagger})}, \quad (6.4)$$

where $\Omega = \Omega_+^\dagger \Omega_-$ and $\Lambda_\pm = \text{diag}(\lambda_1^\pm, \dots, \lambda_N^\pm)$. The Ω integral can be evaluated with help from the Morozov-Eynard formula [18]

$$\int d\Omega \text{tr} \left(\frac{1}{\xi_- + \Lambda_-} \Omega \frac{1}{\xi_+ + \Lambda_+} \Omega^\dagger \right) e^{i\text{tr}(\Lambda_+ \Omega \Lambda_- \Omega^\dagger)} \quad (6.5)$$

$$= \frac{\det \left(S + i \frac{1}{\xi_- + \Lambda_-} S \frac{1}{\xi_+ + \Lambda_+} \right) - \det S}{i\Delta(\lambda^+) \Delta(\lambda^-)},$$

where ξ_\pm are arbitrary numbers and S is a matrix with elements $S_{jk} = \exp(i\lambda_j^+ \lambda_k^-)$.

In a now familiar approach, we will absorb a factor of the Vandermonde determinant into the wavefunction

$$\tilde{\Psi}^\pm(M_\pm) = \Delta(\lambda_\pm) \Psi^\pm(M_\pm) \quad (6.6)$$

as well as write a complete set of wavefunctions in the adjoint sector as

$$\tilde{\Psi}^\pm(M_\pm) = \left(\frac{1}{\xi_\pm + M_\pm} \right) \det_{kl} \left[\psi_{E_k^\pm}^\pm(\lambda_l^\pm) \right], \quad (6.7)$$

where the final factor is a Slater determinant of one-fermion eigenfunctions. In the chiral formalism, the Hamiltonian acts on the wavefunctions (6.6) as

$$H = \mp i \sum_k \left(\lambda_k^\pm \frac{\partial}{\partial \lambda_k^\pm} + \frac{1}{2} \right) \quad (6.8)$$

and the one-fermion eigenfunctions are quite simple

$$\psi^\pm(x^\pm) = \frac{1}{\sqrt{2\pi}} x_\pm^{-\frac{1}{2} \pm iE}. \quad (6.9)$$

The first factor in equation (6.7) is tailored to the use of the Morozov-Eynard formula (6.5). We can interpret equation (6.7) as representing a collective density of eigenvalues given by the Slater determinant together with an impurity wavefunction given by $1/(\xi_\pm + x_\pm)$ for the incoming and outgoing states. We can convert the impurity wavefunction basis to the chiral eigenbasis with the use of this formula:

$$\int_0^\infty d\xi \xi^{iE} \frac{1}{\xi + x} = \frac{i\pi}{\sinh \pi E} x^{iE}. \quad (6.10)$$

The inverse is also going to be useful,

$$\frac{i}{2\xi} \int dE \frac{x^{iE} \xi^{-iE}}{\sinh(\pi E)} = \frac{1}{\xi + x}. \quad (6.11)$$

We wish to compute the in-out overlap $\langle \Psi^- | \Psi^+ \rangle$ as a function of the impurity ‘positions’ ξ_\pm and the occupied energy levels $\{E_k^\pm\}$. In [10], this was computed in the case where $E_k^+ = E_k^- = E_k$, where the energy levels E_k fill the potential up to level μ below the top. That computation gave the scattering phase for the impurity while keeping the Fermi sea in its ground state. We are interested in finding the simplest correction to this scattering involving a change in the Fermi sea. We will therefore take $E_j^- = E_j^+ + \Delta$ for a single

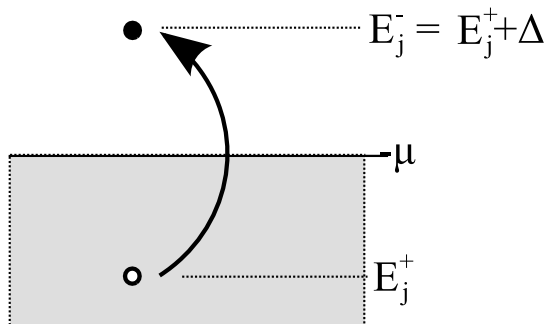


Figure 2: Exciting one fermion above the Fermi sea level at $-\mu$.

fermion labeled by j , and $E_k^+ = E_k^-$ for all $k \neq j$. Choosing to study energy emission from the long string, we take $\Delta > 0$. This will correspond to a single fermion jumping up from the Fermi sea and will be non-zero only if the incoming energy $E_j^+ < -\mu$ and the outgoing energy $E_j^- = E_j^+ + \Delta$ is above the Fermi sea level, $E_j^+ + \Delta > -\mu$. Note that since we are taking μ very large, $E_j^- < 0$, so that nothing spills to the other side of the potential. See figure 2 for a graphical representation.

Let us make a few definitions, following Kostov [10]. The single fermion reflection factor is defined via

$$\int_0^\infty dx_+ dx_- \overline{\psi_E^-(x_-)} \psi_{E'}^+(x_+) e^{ix_+ x_-} = \delta(E - E') R(E) \quad (6.12)$$

and is equal to³

$$R(E) = \int_0^\infty \frac{dr}{\sqrt{r}} r^{iE} e^{ir} = e^{i\pi/4} e^{-\frac{\pi}{2}E} \Gamma\left(\frac{1}{2} + iE\right). \quad (6.13)$$

We also define

$$K(E', E) \equiv \frac{\int_0^\infty dx_+ dx_- \overline{\psi_E^-(x_-)} \psi_{E'}^+(x_+) \frac{e^{ix_+ x_-}}{(\xi_+ + x_+)(\xi_- + x_-)}}{\sqrt{R(E)R(E')}}}, \quad (6.14)$$

which can be evaluated to give [10]

$$K(-E + \epsilon, -E - \epsilon) = \frac{\left(\frac{\xi_+^2}{E}\right)^{i\epsilon} - \left(\frac{\xi_-^2}{E}\right)^{-i\epsilon}}{i \sinh(2\pi\epsilon)} \frac{\pi}{\xi_+ \xi_- - E}, \quad (6.15)$$

where $\epsilon \ll E$ and E is positive.

Using the Morozov-Eynard formula, Kostov in [10] shows that any amplitude can be written as a determinant

$$\langle \Psi^-, \xi_-, E_j^- | \Psi^+, \xi_+, E_j^+ \rangle = \quad (6.16)$$

$$-i \det_{nm} (\delta(E_n^- - E_m^+) + iK(E_n^-, E_m^+)) \sqrt{\prod_{k=1}^N R(E_k^+) \prod_{k=1}^N R(E_k^-)}$$

³The ‘phase factor’ in equation (6.13) is not exactly unitary; there exists a small overlap with the wavefunctions on the other side of the potential. Since we are working in the semi-classical approximation, we can ignore this.

In [10], the main result is that if there were no fermions excited (so that $E_k^+ = E_k^- = E_k$ for all k) then,

$$\langle \Psi^-, \xi_- | \Psi^+, \xi_+ \rangle = \left(\prod_k R(E_k) \right) e^{iS(\xi_+ \xi_-)}. \quad (6.17)$$

The first factor is simply the product of all the scattering phases for all the fermions, an overall phase which is not very interesting and which we will ignore. The second phase is for the scattering of the impurity, which depends (by time invariance) only on the combination $\xi_+ \xi_-$, and is related to the phase for scattering in an energy eigenstate via

$$e^{iS(\xi_+ \xi_-)} = -\frac{1}{4\xi_+ \xi_-} \int dE \frac{(\xi_+ \xi_-)^{-iE}}{(\sinh(\pi E))^2} e^{-i\delta^{Adj}(E)}. \quad (6.18)$$

δ^{Adj} is the scattering phase for an impurity in an energy eigenstate, and the explicit expression computed in [10] agrees with the value computed in string theory.

We are interested in a case where $E_k^+ = E_k^-$ for all but one value of k . The dominant contribution from the determinant in (6.17) comes from the term containing the on-the-diagonal entry $K(E_j^-, E_j^+)$ and a $(N-1) \times (N-1)$ determinant over the other fermions for which it is true that $E_k^+ = E_k^-$. The latter factor in the large N limit is the same as the determinant leading to the phase in (6.18). Therefore, up to an overall constant phase, which includes $\prod_k R(E_k)$, we have

$$\langle \Psi^-, \xi_-, E_j^- | \Psi^+, \xi_+, E_j^+ \rangle = K(E_j^-, E_j^+) \sqrt{R(E_j^-)/R(E_j^+)} e^{iS(\xi_+ \xi_-)}. \quad (6.19)$$

We will take the Mellin transform (6.10) to express this in terms of impurity incoming and outgoing energy eigenstates with energies E_{imp}^\pm

$$\begin{aligned} \langle E_{\text{imp}}^-, E_j^- | E_{\text{imp}}^+, E_j^+ \rangle = \\ -\frac{\sinh(\pi E_{\text{imp}}^-) \sinh(\pi E_{\text{imp}}^+)}{\pi^2} \int_0^\infty d\xi_- \xi_-^{iE_{\text{imp}}^-} \int_0^\infty d\xi_+ \xi_+^{iE_{\text{imp}}^+} \langle \Psi^-, \xi_-, E_j^- | \Psi^+, \xi_+, E_j^+ \rangle. \end{aligned} \quad (6.20)$$

We can now substitute (6.18) and (6.19) into the above, and obtain

$$\begin{aligned} \langle E_{\text{imp}}^-, E_j^- | E_{\text{imp}}^+, E_j^+ \rangle = \\ \frac{\sinh(\pi E_{\text{imp}}^-) \sinh(\pi E_{\text{imp}}^+)}{4\pi^2} \sqrt{R(E_j^-)/R(E_j^+)} \int dE \frac{e^{-i\delta^{Adj}(E)}}{(\sinh(\pi E))^2} \\ \times \int_0^\infty d\xi_- \xi_-^{iE_{\text{imp}}^-} \int_0^\infty d\xi_+ \xi_+^{iE_{\text{imp}}^+} (\xi_+ \xi_-)^{-iE-1} K(E_j^+, E_j^+ + \Delta). \end{aligned} \quad (6.21)$$

The double integral in the last line can be evaluated using the explicit formula in equation (6.15) and the substitution $\xi_\pm = \sqrt{\rho} e^{\pm\sigma}$,

$$\begin{aligned} -\frac{\pi}{i \sinh(\pi\Delta)} \int d\xi_- d\xi_+ \xi_-^{i(E_{\text{imp}}^- - E) - 1} \xi_+^{i(E_{\text{imp}}^+ - E) - 1} \frac{\left(\frac{\xi_+^2}{E}\right)^{-i\Delta/2} - \left(\frac{\xi_-^2}{E}\right)^{i\Delta/2}}{\xi_+ \xi_- - \tilde{E}} \\ = \frac{\pi}{i \sinh(\pi\Delta)} \int_0^\infty \frac{d\rho}{\rho} \rho^{ik} \frac{\left(\frac{\rho}{E}\right)^{i\Delta/2} - \left(\frac{\rho}{E}\right)^{-i\Delta/2}}{\rho - \tilde{E}} \int_{-\infty}^{+\infty} d\sigma e^{i(E_{\text{imp}}^+ - E_{\text{imp}}^- - \Delta)\sigma} \\ = \frac{4\pi^3}{\sinh(\pi\Delta)} \delta(E_{\text{imp}}^+ - \Delta - E_{\text{imp}}^-) \tilde{E}^{ik-1} \Theta(\Delta/2 - |k|), \end{aligned} \quad (6.22)$$

where $\tilde{E} \equiv -(E_j^+ + E_j^-)/2 > 0$, and $k = (E_{\text{imp}}^+ + E_{\text{imp}}^-)/2 - E$. Notice the delta-function, which enforces energy conservation. Plugging this back in,

$$\begin{aligned} \langle E_{\text{imp}}^-, E_j^- | E_{\text{imp}}^+, E_j^+ \rangle &= \delta(E_{\text{imp}}^- + \Delta - E_{\text{imp}}^+) \quad (6.23) \\ &\times \pi \sinh(\pi E_{\text{imp}}^-) \sinh(\pi E_{\text{imp}}^+) \sqrt{R(E_j^-)/R(E_j^+)} \\ &\times \int_{E_{\text{imp}}^-}^{E_{\text{imp}}^+} dE \frac{e^{-i\delta^{Adj}(E)}}{(\sinh(\pi E))^2} \left(-\frac{E_j^+ + E_j^-}{2} \right)^{i((E_{\text{imp}}^+ + E_{\text{imp}}^-)/2 - E) - 1} . \end{aligned}$$

To compare this answer with the string theory amplitude, the outgoing fermionic excitation has to be bosonized. This is relatively simple to do. Start with the standard bosonization formula for a relativistic fermion field $\Psi_R(\sigma)$

$$: \Psi_R^\dagger(\sigma) \Psi_R(\sigma) : = -\frac{1}{\sqrt{\pi}} \partial_\sigma \eta \quad (6.24)$$

and take a Fourier transform

$$\alpha(\Delta) \equiv i\sqrt{2}\Delta \int d\sigma e^{i\sigma\Delta} \eta(\sigma) = \sqrt{2\pi} \int_{-\Delta/2}^{\Delta/2} \frac{d\omega}{2\pi} : \Psi_R^\dagger(\omega + \Delta/2) \Psi_R(\omega - \Delta/2) : . \quad (6.25)$$

When acting on the fermion vacuum, the integrand is nonzero only if $\omega - \Delta/2 < 0$ and $\omega + \Delta/2 > 0$, which is reflected in the choice of range of integration. For $\Delta \neq 0$, we can drop the normal ordering.

It is a well known fact [17] that the fermionic eigenvalues in the matrix model can be treated approximately as relativistic when they are close to the Fermi surface. The fermion creation and destruction operators stay the same, but their interpretation changes: for $\nu > 0$ $\Psi_R(\nu)^\dagger$ used to create a fermion with energy ν , it now creates one with energy $-\mu + \nu$, and $\Psi_R(-\nu)$ used to create an anti-fermion with energy $-\nu$, it now creates a hole in the Fermi sea at energy $\mu - \nu$. Therefore, writing (6.25) in terms of the non-relativistic fermions in section 2 we get

$$\alpha(\Delta) = \sqrt{2\pi} \int_{-\Delta/2}^{\Delta/2} \frac{d\omega}{2\pi} : \Psi(-\mu + \omega + \Delta/2)^\dagger \Psi(-\mu + \omega - \Delta/2) : \quad (6.26)$$

Notice that the fermion bilinear accomplishes exactly the emission process shown in figure 1, with $E_j^+ = -\mu + \omega - \Delta/2$ and $E_j^- = -\mu + \omega + \Delta/2$.

Combining equation (6.26) with equation (6.23), we get, up to an overall phase, the amplitude for emission of a single quantum of the collective field with energy Δ

$$\begin{aligned} \mathcal{A}(E_{\text{imp}}^-, E_{\text{imp}}^+, \Delta) &= \sqrt{\frac{\pi}{2}} \delta(E_{\text{imp}}^- + \Delta - E_{\text{imp}}^+) \sinh(\pi E_{\text{imp}}^-) \sinh(\pi E_{\text{imp}}^+) \\ &\times \int_{-\Delta/2}^{\Delta/2} d\omega \sqrt{R(-\mu + \omega + \Delta/2)/R(-\mu + \omega - \Delta/2)} \quad (6.27) \\ &\times \int_{E_{\text{imp}}^-}^{E_{\text{imp}}^+} dE \frac{e^{-i\delta^{Adj}(E)}}{(\sinh(\pi E))^2} (\mu - \omega)^{i((E_{\text{imp}}^+ + E_{\text{imp}}^-)/2 - E) - 1} . \end{aligned}$$

7. Conclusions and future directions

Equations (3.22) (or (5.14)) and (6.27) constitute the main results of this paper. (5.14) describes interactions between the tip of the long string and the background collective field. These interactions lead to a backreaction on the collective field from the long string scattering. The simplest amplitude induced by such interaction, where one quantum of the collective field is excited, is given in equation (6.27). As has already been mentioned in the introduction, it would be interesting to compare this amplitude with the corresponding disk three-point function in boundary Liouville theory. Agreement would strengthen the conjecture that long strings correspond to the adjoint sector of the matrix quantum mechanics.

Further analysis of the amplitude in (6.27) might help us understand the following puzzle. In the matrix model, the presence of a long string is associated with an impurity, which is a point-like object. The long string itself, however, is a space-filling object, and its interactions with the tachyon field should reflect that somehow. It would be interesting to explore how these two opposing points of view are reconciled.

It would also be interesting to extend our analysis to multiple quanta of tachyon excitations. Potentially, this might lead to a better understanding of the off-shell degrees of freedom in spacetime physics (such as the graviton). The long string should gravitate, and a scattering amplitude involving an incoming and an outgoing tachyon might, for example, contain a recognizable piece due to scattering from its gravitational field. A related approach would be to try to solve equations of motion implied by (5.10), (5.11), (5.13) and (5.14) together, using perturbation theory in μ^{-1} , as was originally developed in [2–4], with the impurity acting as a source for the tachyon field.

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